Math 246C Lecture 3 Notes

Daniel Raban

April 5, 2019

1 Open Mapping, Maximum Principle, Covering Spaces, and Lifts

1.1 The open mapping and the maximum principle

Last time, we showed a local normal form for holomorphic functions:

Proposition 1.1 (local normal form for $f \in \mathrm{Hol}(X, Y)$)**.** *Let X, Y be Riemann surfaces, and let $f_j \in \mathrm{Hol}(X, Y)$ be non-constant. Let $a \in X$. Then there exist complex charts $\varphi : U \to V$ on X with $a \in U$, $\varphi(a) = 0$ and $\psi : U' \to V$; on Y with $f(a) \in U'$, $\psi(f(a)) = 0$, $U \subseteq f^{-1}(U')$ such that the holomorphic function*

$$F = \psi \circ f \circ \varphi^{-1} : V \to V'$$

is of the form $F(z) = z^k$ for some $k \in \mathbb{N}^+$. The integer k is independent of the choice of charts.

Definition 1.1. The integer k is sometimes called the **multiplicity** of f at a. If $k = k(a) > 1$, then a is called a **ramification point**.

Corollary 1.1. *$f \in \mathrm{Hol}(X, Y)$ has no ramification points if and only if f is a local homeomorphism.*

Proof. For any $x \in X$, there is a neighborhood $U \subseteq X$ such that $f : U \to f(U)$ is a homeomorphism. $\square$

Corollary 1.2 (open mapping theorem)**.** *Let $f \in \mathrm{Hol}(X, Y)$ be non-constant. Then f is open.*

Corollary 1.3 (maximum principle)**.** *Let $f \in \mathrm{Hol}(X, \mathbb{C})$ be non-constant. Then $x \mapsto |f(x)|$ does not attain its maximum.*

Proof. If $\sup_{x \in X} |f(x)| = |f(a)|$ for some a, then $f(X) \subseteq \{|z| \leq |f(a)|\}$. $f(X)$ is open, so $f(X) \subseteq \{|z| > f(a)\}$. $\square$

Remark 1.1. In particular, every holomorphic function on a compact Riemann surface is constant.

1.2 Covering spaces and lifts of mappings

Proposition 1.2. *Let X be a Riemann surface, and let Y be a Hausdorff space with a local homeomorphism $p : Y \to X$. There exists a unique complex structure on Y such that $p : Y \to X$ is holomorphic.*

Proof. Existence: Let $\varphi : U \to V$ be a chart on X such that $p : p^{-1}(U) \to U$ is a homeomorphism. Then $\varphi \circ p : p^{-1}(U) \to V$ is a complex chart on Y. These charts define an atlas. Then p is holomorphic. $\square$

Let X, Y, Z be Hausdorff spaces, let $p : Y \to X$ be a local homeomorphism, and let $f : Z \to X$ be continuous. We want a lift $g : Z \to Y$ of f such that $p \circ g = f$.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

Proposition 1.3 (uniqueness of lifts)**.** *Assume that Z is connected. If g_1, g_2 are lifts of f with $g_1(z_0) = g_2(z_2)$, then $g_1 = g_2$.*

Proof. Let $A = \{z \in Z : g_1(z) = g_2(z)\}$ be closed, and let $z_0 \in A$. A is open: Let $z \in A$, $y \in g_1(z)$. Then there exists a neighborhood V of y such that $p : V \to p(V)$ is a homeomorphism. Let W be a neighborhood of z such that $g_1(w) \subseteq V$, $j = 1, 2$. When $z' \in W$, $p(g_1(z')) = p(g_2(z'))$; p is injective, so $g_1 = g_2$ on W. $\square$

Remark 1.2. Assume that X, Y, Z are Riemann surfaces with both p and f holomorphic. Let $\tilde{f} : Z \to Y$ be a lift of f. Then $\tilde{f}$ is holomorphic: $p \circ \tilde{=} f$, where p is a local biholomorphism, so we can locally invert it to get holomorphy of $\tilde{f}$.

Definition 1.2. Let X, Y be topological spaces. A continuous map $p : Y \to X$ is a **covering map** if for all $x \in X$, there is a neighborhood $U \subseteq X$ such that $p^{-1}(U)$ is of the form $p^{-1}(U) = \bigcup_{k \in K} V_k$, where the V_k are open, disjoint, and $p|_{V_k} : V_k \to U$ is a homeomorphism for all k. We say that U is **evenly covered** by p.

Example 1.1. The function $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ given by $z \mapsto e^z$ is a covering map.

Example 1.2. Let Λ be a lattice in $\mathbb{C}$. The projection map $\mathbb{C} \to \mathbb{C}/\Lambda$ is a covering map.

Theorem 1.1. *Let $p : Y \to X$ be a covering map, and let $\gamma : [0, 1] \to X$ be a curve (continuous map) in X. Then for any $y \in p^{-1}(\gamma(0))$, there is a unique lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = y$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

Proof. Consider the open cover of $[0,1]$ by sets of the form $\gamma^{-1}(U)$, where $U \subseteq X$ is evenly covered. There exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open sets $U_k \subseteq X$, $1 \leq k \leq n$ evenly covered by p such that $\gamma([t_{k-1}, t_k]) \subseteq U_k$ for all k (use the existence of a Lebesgue number of the cover). Arguing inductively, assume that we have already constructed a lift $\tilde{\gamma}$ of $[0, t_{k-1}]$, where $k \geq 1$. We have that $p \circ \tilde{\gamma} = \gamma$ on $[0, t_{k-1}]$. In particular, $\tilde{\gamma}(t_{k-1}) \in p^{-1}(U_k) = \bigcup_j V_{k_j}$. So $\tilde{\gamma}(t_{k-i}) \in V_{k_j}$ for some j. We set $\tilde{\gamma}(t) = (p|_{V_{k_j}})^{-1} \circ (\gamma(t))$ for $t_{j-1} \leq t \leq t_k$,s thus lifting $\tilde{\gamma}$ defined on $[0, t_k]$. The uniqueness follows. $\square$

Next time, we will show the existence of universal covering spaces that are simply connected. Eventually, we will show that there are only three such simply connected Riemann surfaces.